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Steady two-dimensional flow of fluid of variable density over an obstacle

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A model of airflow over a mountain is treated mathematically in this paper. The fluid is inviscid, incompressible and of variable density. The flow is in a long channel, bounded above by a rigid horizontal lid and below by an obstacle. The variation with height of the horizontal velocity and of the density is specified far upstream. The details of flow are examined for particular conditions upstream which lead to a linear vorticity equation, although the non-linear inertial terms in the Euler equations of motion are exactly represented. In this case the flow is described by the superposition of solutions of some diffraction problems. Classical techniques of diffraction theory are then used to demonstrate the existence and some general properties of solutions for steady flow. Thus a steady solution is always possible if no restriction is placed on the amount of energy available to drive the flow, that is to say there is no critical internal Froude number (measuring the dynamical effect of buoyancy) for the existence of a steady flow. Finally the flows past a dipole and a vertical wall are computed.

1. Introduction

In a series of papers Long (1953, 1954, 1955) discussed the steady two-dimensional flow of a horizontal stream of variable speed and density over an obstacle. He supposed the fluid inviscid and incompressible. In particular, Long proved that for certain variations with height of the speed and density in the incident stream‡ the governing partial differential equation becomes the reduced wave equation, i.e. Helmholtz's equation. This important discovery made possible exact analysis of stratified flow over obstacles of finite height, though difficulties familiar in diffraction theory make computations non-trivial. Moreover, a vital difference from diffraction theory arises from the boundary conditions at large distances from the obstacle. Long assumed, though this has not been universally accepted (cf. Trustrum 1964), that the horizontal flow is undisturbed far upstream of the obstacle. This 'lee-wave' condition replaces the familiar radiation condition and prevents direct application of diffraction theory.

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‡ More distributions which lead to linear equations such as the reduced wave equation have been found by Long (1958) and Yih (1960).

Previous workers, following Long, have used an inverse method by which they constructed solutions and then replaced suitable streamlines by obstacles. This method has the drawback that the obstacle obtained depends on the stratification of the incident stream, so that one cannot study the dependence on the stratification of the disturbance due to a given obstacle. Further, it is not clear that the flow round any given obstacle with given stratification can be constructed in this way, even in principle.

It seems desirable to modify the powerful computational methods of diffraction theory to find flows over specific obstacles. This is particularly important when the stratification is not small, for then it is difficult even to approximate the desired profile of an obstacle. When these computational methods are inadequate, it is useful to argue about the existence and general properties of solutions for flow over an obstacle by appeal to the well-known physics of diffraction theory. Indeed, the analogy between flow over an obstacle and diffraction theory might lead to some acoustic or radio experiment to model airflow over a mountain.

Long (1955) questioned the existence of solutions, suggesting that if the stratification exceeded a certain critical value then solutions free of upstream waves could occur only for sufficiently small obstacles. Thus for larger obstacles the disturbance might extend far upstream. In effect, Long proposed that intense stratification may give rise to 'blocking' of the flow below the top of the obstacle. This contrasts with the idea of Sheppard (1956), who used Bernoulli's theorem to suggest that blocking would occur if the incident stream had insufficient energy. The idea of blocking has been further examined by Yih (1959) for two-dimensional flow, and by Drazin (1961) for three-dimensional flow.

In §§2, 3 we define the model and state the equation of motion and the boundary conditions. The problem for any given obstacle of finite height is treated generally in §4. We reduce the problem to an equivalent set of problems of classical diffraction theory in a wave-guide. Our arguments suggest, in contradiction to Long's, that the problem is well posed except for an enumerable infinity of cases of 'resonance'. In §5 the particular case of a line dipole on the bed of the channel is treated. The solution is not for various incident streams past a prescribed obstacle, but it has the merit of simplicity. An explicit analytic solution is found, which gives an explicit expression for the wave-drag and facilitates computation of the streamlines. The more important problem of prescribing an obstacle and then, for all values of the internal Froude number of the incident stream, determining the flow is solved in §6 for a particular simple obstacle, namely the vertical strip. The analysis is numerical. In §7 we relate our work to that of other authors, and in particular we try to meet Mrs Trustrum's (1964) criticisms of the boundary conditions upstream.

We stress that our work is mathematical. We have not re-examined the mechanical postulates of the theory, working within the framework of Long's model and its extension (Scorer & Klieforth 1959) to the case of flow with closed streamlines, i.e. with rotors. This reveals more about stratified flows, in particular about blocking.

Recent work has shown that Long's model whereby the governing equation of the flow is linear is special in some sense. Benjamin (1966) and Long (1965) himself have shown that this model permits neither a solitary wave nor a hydraulic jump. However, these are properties of the flow at large, relating the flow far upstream with that far downstream. The generation of finite lee waves and rotors is a local property, so it is plausible that the qualitative nature of such flow near an obstacle does not depend crucially on the special nature of the incident stream.

2. Equation of motion

We consider two-dimensional steady flow of inviscid incompressible fluid of variable density. Far upstream the fluid has velocity $U_{-\infty}(y)\mathbf{i}$ and density $\rho_{-\infty}(y)$, varying with height y in a prescribed way, where the unit vector $\mathbf{i}$ is horizontal and is in the direction of the x-axis. This stream runs in a long channel whose upper boundary is a rigid horizontal plane $y = T$ and whose lower boundary is the rigid obstacle $y = h(x)$, where $h(x) \to 0$ as $x \to \pm\infty$.

Long (1953) used as dependent variable the height far upstream $y_{-\infty}(x, y)$ of the streamline through the point (x, y). He found the equation governing $y_{-\infty}$ in regions where the streamlines come from far upstream. In the special case

$$U^2_{-\infty}\rho_{-\infty} = \text{const.}, \quad d\rho_{-\infty}/dy = -\beta\rho_0, \tag{2.1}$$

where β and ρ_0 are constants, he found $y_{-\infty}$ satisfies the linear equation

$$\nabla^2 y_{-\infty} + g\beta\rho_0(y_{-\infty} - y)/U^2_{-\infty}\rho_{-\infty} = 0. \tag{2.2}$$

Now let us introduce the dimensionless variables

$$\mathbf{r}' = \pi\mathbf{r}/T, \quad h'(x') = \pi h(x)/T, \quad \delta' = \pi(y - y_{-\infty})/T. \tag{2.3}$$

In these variables equation (2.2) becomes the reduced wave-equation,

$$\nabla'^2\delta' + k^2\delta' = 0, \tag{2.4}$$

where

$$k^2 \equiv g\beta\rho_0 T^2/\pi^2 U^2_{-\infty}\rho_{-\infty} \tag{2.5}$$

is an internal Froude number. For the incident stream to be stable we must have $\beta \geqslant 0$ and therefore $k^2 \geqslant 0$.

Clearly $y_{-\infty}(x, y) = \text{const.}$ is a streamline, so that the boundary conditions on the top and bottom of the channel are

$$\delta' = 0 \quad \text{for} \quad y' = \pi, \tag{2.6}$$

$$\delta' = h'(x') \quad \text{for} \quad y' = h'(x'). \tag{2.7}$$

In §3 we shall apply the condition that there are no waves far upstream.

Finally we note that the wave resistance, which arises from the flux of momentum in the waves downstream, can be shown by a short calculation to be

$$D = (\tfrac{1}{2}\rho_{-\infty} U^2_{-\infty}T/\pi)\int_0^\pi [(\partial\delta'/\partial x')^2 - (\partial\delta'/\partial y)^2 + k^2\delta'^2]_{x'=+\infty}\, dy' \tag{2.8}$$

for the special case (2.1).

3. The upstream boundary condition

We have seen that we have to solve the reduced wave equation, and so seek to apply the well-established techniques of diffraction theory to the present problem. However, many of these techniques depend intimately on the radiation condition (cf. Noble 1958, p. 27) and therefore cannot be applied directly. In this section we show what the lee-wave condition means mathematically, and in the next we show how it can be satisfied if we can solve a certain finite set of classical diffraction problems. Not only will this open the problem to well-known techniques, but it also enables us to think physically about the situation, since there is a reliable physical picture of a diffraction problem, even when a detailed mathematical solution is not available.

We first consider a channel with no obstacle present, that is with $h(x) \equiv 0$. (We drop the dashes from dimensionless quantities henceforth.) Let $\delta(x,y)$ be any solution of the equation

$$\nabla^2\delta + k^2\delta = 0 \tag{3.1}$$

with

$$\delta = 0\,(y = 0, \pi). \tag{3.2}$$

By Fourier analysis it follows that

$$\delta = \sum_{n=1}^{\infty} \delta_n(x)\sin ny, \tag{3.3}$$

where

$$d^2\delta_n/dx^2 + (k^2 - n^2)\,\delta_n = 0. \tag{3.4}$$

Clearly a lot depends on the sign of $(k^2 - n^2)$. If $0 \leqslant k < 1$, then for each value of n equation (3.4) has exponential solutions,

$$\delta_n = A_n' \exp\{(n^2 - k^2)^{\frac{1}{2}}x\} + A_n \exp\{-(n^2 - k^2)^{\frac{1}{2}}x\}, \tag{3.5}$$

where A_n', A_n are arbitrary constants. There is no solution bounded in the whole channel other than $\delta \equiv 0$; indeed, this is why there is no solitary wave for this model (Long 1965). The structure of the flow is similar to that in the case $k = 0$ of potential flow. In hydraulic language, the flow is supercritical when $k < 1$.

If $K < k < K+1$ for some positive integer K, equation (3.4) also has K solutions of sinusoidal type,

$$\delta_n = A_n' \sin\{(k^2 - n^2)^{\frac{1}{2}}x\} + A_n \cos\{(k^2 - n^2)^{\frac{1}{2}}x\} \quad (1 \leqslant n \leqslant k). \tag{3.6}$$

These are stationary waves, and in hydraulic language the flow is subcritical.

The cases where k is an integer are singular, and correspond to wave-guide resonance between the walls $y = 0, \pi$. In hydraulic language, the flow is critical. We can, by analogy, anticipate that there will be no steady flow in these cases, so that we can specifically exclude them by demanding that k be not an integer.

This analysis of the unobstructed channel throws light on that of the channel obstructed by a finite obstacle which has $h(x) = 0$ for $|x| > L$. (This assumption that $h(x)$ is of compact support is not necessary, for all we need is that $h(x) \to 0$ sufficiently rapidly as $x \to \pm\infty$, but it is convenient to make it to simplify the formulation.) The above analysis then applies for $|x| > L$; for $|x| < L$ we must find the general solution by other means and match along the lines $x = \pm L$.

If $0 \leqslant k < 1$, the obvious requirement that δ be bounded means that $A'_n = 0$ $(x > L)$ and $A_n = 0\,(x < -L)$. These severe restrictions on the form of the solution lead us to anticipate that there is a unique bounded solution when $0 \leqslant k < 1$; indeed, we know from potential theory that this is true when $k = 0$. The uniqueness of the solution for $0 < k < 1$ can be proved by standard methods (cf. Couran- & Hilbert 1953) when $h(x) \geqslant 0$. Thus the case $0 \leqslant k < 1$ closely resembles the harmonic case $k = 0$, and we need no extra boundary conditions. We should expect, by analogy with supercritical flow in hydraulics, that the wave resistance is zero. In fact this follows from equation (2.8) and the disturbance's vanishing exponentially at infinity.

If $k > 1$, we are on less familiar ground. Following Long (1953), we take the solution free of waves far upstream of the obstacle; that is, we require

$$A_n, A'_n = 0 \quad (x < -L, \quad 1 \leqslant n \leqslant K). \tag{3.7}$$

Two questions at once arise. Does a solution exist at all? If it does, are conditions (3.7) enough to make it unique? Long (1955, p. 344) suggested that conditions (3.7) could be too restrictive, and that if the obstacle were too high no solution at all could be found. This is plausible, and offers a dynamical rather than an energetic explanation of blocking. However, Long's argument implicitly assumes that δ is regular not only in $h(x) \leqslant y \leqslant \pi$ but also in $0 \leqslant y \leqslant h(x)$, and this assumption may be false. To see this, we recall that Yih (1960) has constructed obstacles by placing dipoles in the region $0 < y < h(x)$. Our original aim was to place this result of Long on a firmer footing, but instead we shall, in §4, show it to be mistaken.

4. Reduction to a set of diffraction problems

We want to find a function $\delta(x, y)$ which satisfies the reduced wave equation

$$\nabla^2\delta + k^2\delta = 0, \tag{4.1}$$

the boundary conditions on the rigid walls,

$$\delta = 0 \quad \text{for} \quad y = \pi, \quad \delta = h(x) \quad \text{for} \quad y = h(x), \tag{4.2}$$

and the conditions at infinity,

$$\delta \text{ is bounded as } x \to +\infty, \tag{4.3}$$

$$\delta \to 0 \text{ exponentially as } x \to -\infty. \tag{4.4}$$

Let us consider the related problem of finding the solution $\delta(x, y, t)$ of the scalar wave equation

$$(\nabla^2 - \partial^2/\partial t^2)\delta = 0$$

which satisfies the conditions

$$\delta = 0 \quad \text{for} \quad y = \pi,$$
$$\delta = h(x)\, e^{ikt} \quad \text{for} \quad y = h(x).$$

In physical terms of an acoustic problem, δ might now be the pressure in a gas filling the channel with a 'soft' top (one on which the pressure is constant) and a constructed bottom $y = h(x)$ on which the pressure is a prescribed small harmonic function of time. Then the assumption of the time factor e^{ikt} for $\delta(x,y,t)$ leads to the problem (4.1), (4.2), but with conditions (4.3), (4.4) replaced by the Sommerfeld radiation condition. Continuing the solution of this acoustic problem, we suppose K to be the integer such that $K < k < K+1$ and introduce the notation

$$\left.\begin{aligned}\overleftarrow{W_n}, \overrightarrow{W_n} &= \exp\{\pm i(k^2-n^2)^{\frac{1}{2}}x\}\sin ny \quad (1 \leqslant n \leqslant K),\\ \overleftarrow{E_n}, \overrightarrow{E_n} &= \exp\{\pm (n^2-k^2)^{\frac{1}{2}}x\}\sin ny \quad (K+1 \leqslant n).\end{aligned}\right\} \tag{4.5}$$

Then the Sommerfeld radiation condition requires that the solution δ_e of this emission problem has the properties

$$\delta_e = \sum_{n=1}^{K} A_n \overleftarrow{W_n} + \sum_{n=K+1}^{\infty} A_n \overleftarrow{E_n} \quad \text{when} \quad x < -L, \tag{4.6}$$

$$\delta_e = \sum_{n=1}^{K} A'_n \overrightarrow{W_n} + \sum_{n=K+1}^{\infty} A'_n \overrightarrow{E_n} \quad \text{when} \quad x > L, \tag{4.7}$$

for some constants A_n, A'_n. If we consider δ_e in the light of conditions (4.3), (4.4) we see that it is at once too free and too restrictive. It has waves

$$\sum_{n=1}^{K} A_n \overleftarrow{W_n}$$

in the region $x < -L$ where they are forbidden by (4.4). On the other hand, the radiation condition eliminates $\overleftarrow{W_n}$, which represents incoming radiation, and condition (4.3) only requires boundedness for $x > L$. Without violating condition (4.3), we can place a wave generator at $x = +\infty$ to produce a linear combination

$$\sum_{n=1}^{K} \epsilon_n \overleftarrow{W_n}$$

of incoming radiation for $x > L$. These waves will impinge on the constriction and be both reflected and transmitted. We aim to choose the constants ϵ_n (that is, to adjust the wave generator) so that the transmitted radiation just cancels the emitted radiation

$$\sum_{n=1}^{K} A_n \overleftarrow{W_n} \quad \text{where} \quad x < -L.$$

More precisely, we consider K reflexion problems for which we seek solutions $\delta_f^{(n)}$ $(n = 1, 2, \ldots, K)$ of the reduced wave equation (4.1) and the conditions

$$\delta_f^{(n)} = 0 \quad \text{for} \quad y = \pi, \tag{4.8}$$

$$\delta_f^{(n)} = 0 \quad \text{for} \quad y = h(x), \tag{4.9}$$

$$\delta_f^{(n)} \sim \overleftarrow{W_n} + \text{outgoing radiation} \quad \text{as } x \to +\infty, \tag{4.10}$$

$$\delta_f^{(n)} \sim \text{outgoing radiation} \quad \text{as } x \to -\infty. \tag{4.11}$$

These K solutions $\delta_f^{(n)}$ have the properties

$$\delta_f^{(n)} = \sum_{r=1}^{K} D_r^{(n)} \overleftarrow{W_r} + \sum_{r=K+1}^{\infty} D_r^{(n)} \overleftarrow{E_r} \quad \text{when} \quad x < -L, \tag{4.12}$$

$$\delta_f^{(n)} = \overleftarrow{W_n} + \sum_{r=1}^{K} C_r^{(n)} \overrightarrow{W_r} + \sum_{r=K+1}^{\infty} C_r^{(n)} \overrightarrow{E_r} \quad \text{when} \quad x > L. \tag{4.13}$$

Clearly, if we can choose the constants ϵ_n so that

$$\sum_{n=1}^{K} \epsilon_n \left\{ \sum_{r=1}^{K} D_r^{(n)} \overleftarrow{W_r} \right\} + \sum_{r=1}^{K} A_r \overleftarrow{W_r} = 0, \tag{4.14}$$

then

$$\delta = \sum_{n=1}^{K} \epsilon_n \delta_f^{(n)} + \delta_e$$

will satisfy the lee-wave problem (4.1)–(4.4), which we have been aiming to solve. So we must now examine whether this procedure will work.

We shall assume, on physical grounds, that solutions δ_e and $\delta_f^{(n)}$ to the emission and K reflexion problems exist. But are these solutions unique? It is easy to see that they are not unique if and only if eigensolutions for the channel exist; that is, there exist functions δ satisfying (4.1), (4.8), (4.9) and the additional condition

$$\delta \to 0 \quad \text{as} \quad x \to \pm\infty. \tag{4.15}$$

The existence of eigensolutions for a domain of given shape is a deep question (cf. Müller 1957) and we shall not try to answer it rigorously. However, because an eigensolution represents radiation trapped by an obstacle, it seems unlikely on physical grounds that one occurs for an obstacle with no cleft. Thus one anticipates that trapping is associated with an obstacle having a cavity with a narrow outlet, that is an obstacle whose normal somewhere makes more than a right angle with the vertical. On these grounds we assume uniqueness of our solution.

Since the functions $\overleftarrow{W_r}$ are linearly independent, equation (4.14) is equivalent to the system of equations

$$\sum_{n=1}^{K} \epsilon_n D_r^{(n)} + A_r = 0 \quad (r = 1, 2, \ldots, K). \tag{4.16}$$

Therefore the success of our method, and hence the existence of a solution of the lee-wave problem (4.1)–(4.4), is equivalent to the solubility of this system (4.16), granted the existence of δ_e and $\delta_f^{(n)}$. Thus the necessary and sufficient condition for the problem (4.1)–(4.4) to have a solution is that

$$\det (D_r^{(n)}) \neq 0. \tag{4.17}$$

If a solution exists it will, under our assumption, be unique.

The physical significance of condition (4.17) is seen most clearly in the case $K = 1$ when there is only one wave-like solution possible at infinity. Then condition (4.17) gives

$$D_1^{(1)} \neq 0, \tag{4.18}$$

that is the transmission coefficient of the constricted channel does not vanish. This seems to be obvious physically, even if $h_{\max}$ is only slightly less than π. We confirm this by actual calculation in a special case in § 6. For $K > 1$, condition (4.17) implies that no linear combination of incoming radiation can be totally reflected.

5. Dipole solution

We shall see that formidable difficulties stand in the way of an explicit solution of the wave-guide diffraction problems posed in the last section. Before considering the simplest such problem in § 6 we have, at the suggestion of Dr L. E.

Fraenkel, considered the disturbance to the flow caused by a dipole in the bottom of the channel. Our analysis is similar to that Fraenkel (1956) used for rotating fluids. Our problem has also been proposed by Yih (1960, p. 170). Its analogy in three dimensions for an open channel ($T = \infty$) has been solved by Crapper (1959).

We regard the solution analytically, that is as a fundamental solution rather than as a practical solution for a real dipole. Thus we consider a solution of the vorticity equation (4.1) which has a prescribed dipole singularity rather than solutions for dipoles emitting various distributions of density and vorticity. The solution can then be used to give the flow over any obstacle that happens to coincide with any of the streamlines. This inverse method of finding $y = h(x)$ to fit the solution $\delta(x,y)$ has the grave disadvantage that the shape and size of the streamlines change as the internal Froude number k^2 changes—whereas the chief interest is how the flow over a prescribed obstacle changes with k^2. Nevertheless, the dipole solution is not without interest and does give some information as to the drag and to the nature of the flow at large k^2.

A solution of the reduced wave equation (4.1) representing a dipole at the origin with horizontal axis is

$$\delta = \tfrac{1}{2}ik\mu \sin\theta H_1^{(1)}(kr),$$

where $r \equiv (x^2+y^2)^{\frac{1}{2}}$, $\theta \equiv \tan^{-1}(y/x)$ and μ is any real constant. The choice of the Hankel function $H_1^{(1)}$ is dictated by the radiation condition; however, relaxing this condition on the solution does not affect either the singularity at the origin or the final real solution, which are our concern here. In fact this dipole solution $\delta(x,\, y)$ behaves like $\mu\delta(x)$ on $y = 0$; where $\delta(x)$ is the Dirac delta-function. Thus, to find the disturbances to the flow with this dipole placed on the channel floor at the origin, we have to solve the following problem:

$$\nabla^2\delta + k^2\delta = 0 \quad (r \neq 0), \tag{5.1}$$

$$\delta = 0 \quad \text{for} \quad y = \pi, \tag{5.2}$$

$$\delta = \mu\delta(x) \quad \text{for} \quad y = 0, \tag{5.3}$$

$$\delta \text{ bounded as } x \to +\infty, \tag{5.4}$$

$$\delta \to 0 \quad \text{as} \quad x \to -\infty. \tag{5.5}$$

This lee-wave problem can be solved by the method described in the last section. However, it is such a simple example that it is possible to satisfy conditions (5.4), (5.5) directly by use of a method of Lamb (1932, §§242–6) for surface waves.

It can be seen that

$$\delta(x,y) = \frac{\mu}{2\pi}\int_\Gamma e^{i\alpha x}\frac{\sinh\lambda(\pi-y)}{\sinh\lambda\pi}\,d\alpha \tag{5.6}$$

satisfies the equation (5.1) and the boundary conditions (5.2), (5.3) where† $\lambda \equiv (\alpha^2-k^2)^{\frac{1}{2}}$ and the contour Γ goes somehow from the left end to the right end of the real α-axis in the complex α-plane. When $y \neq 0$ or π, singularities occur as simple poles at the zeros of $\sinh\lambda\pi$, i.e. at $\alpha = \pm(k^2-n^2) = \pm\alpha_1,\ \pm\alpha_2, \ldots,$ $\pm\alpha_K$(real), $\pm\alpha_{K+1},\ \pm\alpha_{K+2}, \ldots$ (pure imaginary). Thus boundary conditions

† To make λ single-valued we may place a cut between the branch points $\alpha = -k$ and $\alpha = +k$ and require that λ be positive on the real α-axis where $\alpha > k$.

(5.4), (5.5) can be satisfied by taking Γ just below the real axis in the α-plane. Lamb's method is to evaluate δ by the calculus of residues, by closing Γ at infinity in the upper half-plane for $x > 0$, and in the lower half-plane for $x < 0$. In this way one can show that

$$\delta(x,y) = \begin{cases} \dfrac{\mu}{\pi} \displaystyle\sum_{n=K+1}^{\infty} \dfrac{\sin ny \exp\{(n^2-k^2)^{\frac{1}{2}}x\}}{(1-k^2/n^2)^{\frac{1}{2}}} & \text{when} \quad x \leqslant 0, \\ \dfrac{\mu}{\pi}\left[-2\displaystyle\sum_{n=1}^{K} \dfrac{\sin ny \sin\{(k^2-n^2)^{\frac{1}{2}}x\}}{(k^2/n^2-1)^{\frac{1}{2}}} \right. & \\ \left. \quad + \displaystyle\sum_{n=K+1}^{\infty} \dfrac{\sin ny \exp\{-(n^2-k^2)^{\frac{1}{2}}x\}}{(1-k^2/n^2)^{\frac{1}{2}}}\right] & \text{when} \quad x \geqslant 0. \end{cases} \tag{5.7}$$

It is interesting to note that the problem could almost as easily be solved by the method of §5. We have only to solve the emission problem because, in this case, there is no reflexion and we have $D_r^{(n)} = \delta_{nr}$, the Kronecker delta. Further note that when $k^2 = 0$ summation of the series (5.7) gives the potential solution

$$\delta(x,y) = \mu \sin y/[2\pi(\cosh x - \cos y)].$$

We can use the solution (5.7) to calculate the drag on the dipole. Use of formula (2.8) shows that

$$D = \begin{cases} 0 & (0 < k < 1) \\ \frac{1}{12}\pi^2\mu^2\rho_{-\infty}U_{-\infty}^2 TK(K+\frac{1}{2})(K+1) & (K < k < K+1). \end{cases} \tag{5.8}$$

Thus the drag is a step function of k, and increases like k^3 as $k \to \infty$. This also shows that the limit of D is independent of T as $T \to \infty$ for fixed μT^2, in which limit the dimensional solution near the dipole is independent of T.

We can use this solution to construct flows over obstacles by replacing streamlines by solid surfaces. In general this requires numerical computation of (5.7) and, moreover, the shape of the obstacle depends on k. If, however, $r = (x^2+y^2)^{\frac{1}{2}}$ is very small δ is dominated by the dipole singularity, so that

$$\delta \sim \tfrac{1}{2}ik\mu \sin\theta H_1^{(1)}(kr) \quad \text{as} \quad r \to 0.$$

If, in addition, we let $kr \to 0$, then

$$\delta \sim \mu y/\pi r^2. \tag{5.9}$$

Now the boundary condition (4.2) on a rigid surface $y = h(x)$ is that

$$\delta = h(x) \quad \text{for} \quad y = h(x).$$

Clearly this condition is satisfied on the small circle $r = a$ by the approximate solution (5.9) if we choose $\mu = \pi a^2$. This result is *independent* of k so long as $k \ll 1$. Thus we have the disturbance caused by a small semi-circle of fixed size. If A is the dimensional radius, then $a = A\pi/T$ and formula (5.8) shows that the drag is $\pi^4 A^4 \rho_{-\infty} U_{-\infty}^2 K(K+\frac{1}{2})(K+1)/12T^3$.

We have evaluated with an electronic digital computer the solution (5.7) for $\mu = 1$, $k^2 = \frac{1}{2}$, 2 and 12. Some streamlines for the three cases are plotted in figures 1, 2 and 3 respectively. Figure 1 depicts a supercritical flow with no lee waves, the flow being symmetric about the vertical line $x = 0$; the dipole causes only a slight disturbance of the stratified incident stream. Figure 2 depicts a subcritical flow with one lee wave ($K = 1$); the effects of stratification are disturbing the incident stream quite significantly. Figure 3 depicts a subcritical flow with three lee waves superposed ($K = 3$); there are several regions of closed streamlines, in some of which the streamfunction takes values beyond those taken far upstream in the channel; this case indicates that the steady flow becomes very confused as k increases, and probably develops into turbulence.

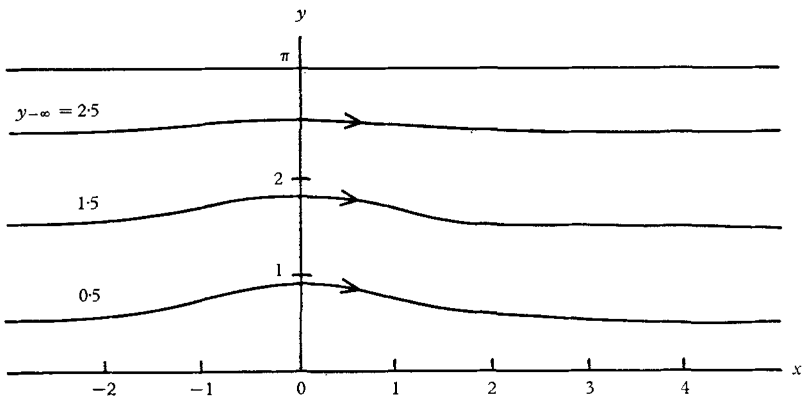

FIGURE 1. Flow past a dipole: $\mu = 1$, $k^2 = \frac{1}{2}$.

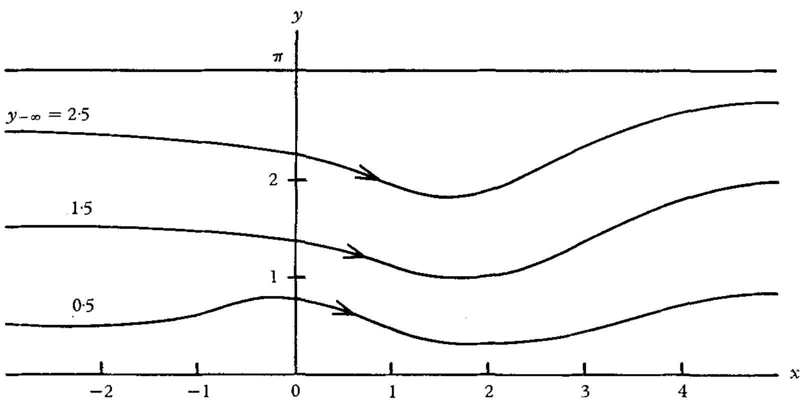

FIGURE 2. Flow past a dipole: $\mu = 1$, $k^2 = 2$.

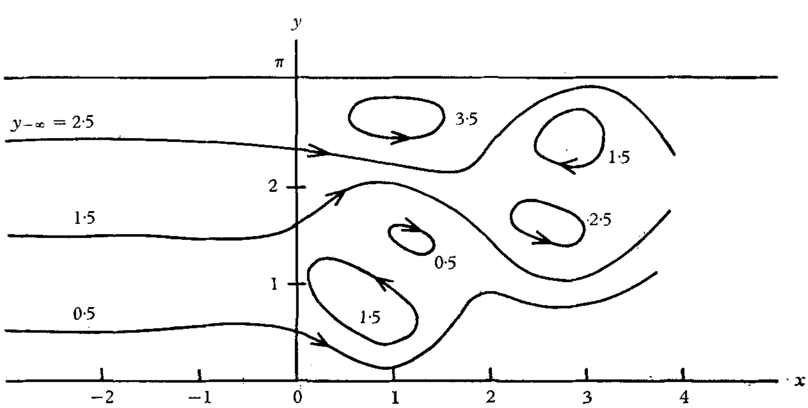

FIGURE 3. Flow past a dipole: $\mu = 1$, $k^2 = 12$.

As a postscript to this section we consider the solution when k is an integer. There the simple poles $\pm\alpha_K$ coalesce at the origin in the α-plane to form a double pole. On evaluation of the residue there, one finds a term which is unbounded like $x \sin Ky$ as $x \to +\infty$. Thus the problem (5.1)–(5.4) has no solution because there is resonance between the walls $y = 0, \pi$ of the wave-guide.

6. The disturbance due to a vertical strip

In the previous section we have examined the disturbance created by a line dipole on the bed of the channel. We saw that, though we could construct a flow past any rigid obstacle coincident with a streamline, the shape of the obstacle could not be prescribed in advance and, more seriously, depended on the value of the internal Froude number. Here we wish to consider the waves set up by a prescribed obstacle so that we can see how the flow pattern varies with k. As we have shown (in § 4) that the determination of the flow field under the 'no-waves-upstream' boundary condition can be reduced to the solution of two types of classical wave-guide diffraction problems, we can seek help in the well-established theory of wave-guides (cf. Jones 1964). For an obstacle of general shape in the wave-guide, diffraction problems are quite intractable unless k is small, in which case an expansion about the (in principle!) known solutions of Laplace's equation in the wave-guide can be used. Examples of this method are given by Lamb (1932). In our case k can be large, however. To make any progress it seems best to simplify the obstacle, so we have decided to confine our attention to an obstacle in the form of a vertical strip. More precisely, if $y = 0$ and $y = \pi$ are the channel walls, the obstacle is the strip $x = 0\,(0 \leqslant y \leqslant d < \pi)$. Several approximate methods to find the reflexion at such an obstacle (called a *capacitative iris*) in a wave-guide have been developed, and the special case with $d = \frac{1}{2}\pi$ has been solved exactly. (Jones (1964) gives a full account and bibliography.)

We first solved this problem by explicitly splitting it up into a superposition of the emission problem and K transmission problems, and then computed streamlines in various special cases below. However, Mrs K. Trustrum has suggested a synthesis of these problems which is equivalent but somewhat shorter and simpler, so this is the method we present below. We seek $\delta(x, y)$ such that

$$\left.\begin{aligned} &\nabla^2\delta + k^2\delta = 0;\\ &\delta = 0 \quad \text{on} \quad y = 0, \pi,\\ &\delta = y \quad \text{on} \quad x = 0, \quad 0 \leqslant y \leqslant d,\\ &\delta \to 0 \quad \text{as} \quad x \to -\infty, \quad \delta \text{ bounded as } x \to +\infty. \end{aligned}\right\} \tag{6.1}$$

In the usual way, we may satisfy the equation and all the boundary conditions except those at $x = 0$ by taking

$$\delta = -2\sum_{r=1}^{K}\{A_r \sin\lambda_r x + B_r \cos\lambda_r x\}\sin ry + \sum_{r=K+1}^{\infty} A_r \exp(-\lambda_r x)\sin ry \quad (x \geqslant 0, \quad 0 \leqslant y \leqslant \pi),$$

$$\delta = \sum_{r=K+1}^{\infty} A'_r \exp(\lambda_r x)\sin ry \quad (x \leqslant 0, \ 0 \leqslant y \leqslant \pi),$$

for some constants A_r, B_r, A'_r, where

$$\lambda_r = \begin{cases} +(k^2-r^2)^{\frac{1}{2}} & (1 \leqslant r \leqslant K), \\ +(r^2-k^2)^{\frac{1}{2}} & (K+1 \leqslant r). \end{cases} \tag{6.2}$$

The boundary conditions at $x = 0$ are that δ and $\partial\delta/\partial y$ be continuous for $d \leqslant y \leqslant \pi$ and that $\delta = y$ for $0 \leqslant y \leqslant d$. Therefore we have

$$A'_r = A_r \quad (r \geqslant K+1),$$

$$B_r = 0 \quad (r \leqslant K),$$

$$\sum_{r=K+1}^{\infty} A_r \sin ry = y \quad (0 \leqslant y \leqslant d),$$

$$-2\sum_{r=1}^{\infty} \lambda_r A_r \sin ry - \sum_{r=K+1}^{\infty} \lambda_r A_r \sin ry = \sum_{r=K+1}^{\infty} \lambda_r A_r \sin ry \quad (d \leqslant y \leqslant \pi).$$

In summary, we have a solution

$$\delta = -(1+\operatorname{sign} x) \sum_{r=1}^{K} A_r \sin \lambda_r x \sin ry + \sum_{r=K+1}^{\infty} A_r \exp(-\lambda_r|x|) \sin ry$$
$$(0 \leqslant y \leqslant \pi, \quad -\infty < x < \infty), \tag{6.3}$$

where

$$\left.\begin{aligned} \sum_{r=K+1}^{\infty} A_r \sin ry &= y \quad (0 \leqslant y \leqslant d), \\ \sum_{r=1}^{\infty} \lambda_r A_r \sin ry &= 0 \quad (d \leqslant y \leqslant \pi). \end{aligned}\right\} \tag{6.4}$$

Unfortunately, dual Fourier series of the type (6.4) cannot in general be solved in closed form to yield the unknown coefficients A_r. The present pair have been solved (Tranter 1959) only in the case of a harmonic field ($k = 0$). Instead, we shall convert equation (6.4) into a single integral equation and then solve that integral equation by a method described by Jones (1964). We do not know the values of the sum

$$g(y) = \sum_{r=1}^{\infty} \lambda_r A_r \sin ry \quad \text{for} \quad 0 \leqslant y \leqslant d;$$

suppose then that we take this function as our unknown rather than the coefficients A_r. Therefore, using the second of equation (6.4) and assuming that $g(y)$ satisfies the conditions of Fourier's theorem,† we have

$$A_r = \frac{2}{\pi\lambda_r}\int_0^d g(y') \sin ry' \, dy'. \tag{6.5}$$

Substitution of (6.5) into the first equation of (6.4) now yields the integral equation

$$\int_0^d g(y') K(y,y') \, dy' = y \quad (0 \leqslant y \leqslant d), \tag{6.6}$$

† Clearly $g(y) = -(\partial\delta/\partial x)_{x=0}$, so we anticipate a singularity like $|y-d|^{-\frac{1}{2}}$ near $y = d$.

where the kernel

$$K(y,y') = (2/\pi) \sum_{r=K+1}^{\infty} \lambda_r^{-1} \sin ry \sin ry' \tag{6.7}$$

$$= (2/\pi) \sum_{r=1}^{\infty} (\lambda_r^{-1} - r^{-1}) \sin ry \sin ry'$$

$$-(2/\pi) \sum_{r=1}^{K} \lambda_r^{-1} \sin ry \sin ry' + (1/\pi) \log|\sin \tfrac{1}{2}(y+y')/\sin \tfrac{1}{2}(y-y')|. \tag{6.8}$$

Owing to the presence of λ_r, the kernel (6.7) cannot be expressed in closed form except in the case $k = 0$. As $r \to \infty$, $\lambda_r^{-1} - r^{-1} = O(r^{-3})$, so the infinite series of (6.8) converges uniformly, whilst the latter term is a series summed to reveal nothing worse than the logarithmic singularity of the kernel at $y = y'$. Thus the integral equation (6.6) is regular.

The method described by Jones is to expand $g(y)$ and y in terms of the complete set of orthogonal functions, $\{\sin (s\pi y/d)\}$ for each integer s. Thus we write

$$g(y) = \sum_{s=1}^{\infty} g_s \sin (s\pi y/d), \tag{6.9}$$

$$y = (2/d) \sum_{s=1}^{\infty} Y_s \sin (s\pi y/d), \tag{6.10}$$

$$\sin ry = (2/d) \sum_{s=1}^{\infty} P_{rs} \sin (s\pi y/d), \tag{6.11}$$

$(0 \leqslant y \leqslant d)$ for g_s to be determined, where

$$Y_s = \int_0^d y \sin (s\pi y/d)\, dy = (-1)^{s+1} d^2/s\pi, \tag{6.12}$$

$$P_{rs} = \int_0^d \sin ry \sin (s\pi y/d)\, dy$$

$$= \begin{cases} \dfrac{(-1)^s s\pi d \sin rd}{(rd)^2 - (s\pi)^2} & (rd \neq s\pi), \\ \tfrac{1}{2}d & (rd = s\pi). \end{cases} \tag{6.13}$$

Now integral equation (6.6) gives

$$\sum_{s,t=1}^{\infty} g_s K_{st} \sin (t\pi y/d) = \sum_{t=1}^{\infty} Y_t \sin (t\pi y/d) \quad (0 \leqslant y \leqslant d)$$

where

$$K_{st} = (2/\pi) \sum_{r=K+1}^{\infty} \lambda_r^{-1} P_{rs} P_{rt}. \tag{6.14}$$

Equating coefficients of the orthogonal functions, we deduce that

$$\sum_{s,t=1}^{\infty} g_s K_{st} = Y_t \quad (t = 1, 2, \ldots). \tag{6.15}$$

As it stands, this infinite system is no more tractable than the integral equation (6.6) from which it was derived. However, we can solve it for g_s approximately but effectively by truncation—that is, we assume $g_{M+1}, g_{M+2}, \ldots = 0$ and keep only the first M of equations (6.15) for some positive integer M. Then we find that equation (6.5) gives

$$A_r = (2/\pi\lambda_r) \sum_{s=1}^{M} g_s P_{rs}. \tag{6.16}$$

We remark how slowly the series (6.16) converges. For we know that $g(y)$ behaves like $|y-d|^{\frac{1}{2}}$ near $y = d$ and from this that $g_s = O(s^{-\frac{1}{2}})$ as $s \to \infty$.† The definition (6.13) shows that $P_{rs} = O(s^{-1})$ as $s \to \infty$ for fixed r. Thus (6.16) converges no faster than terms of order $s^{-\frac{3}{2}}$, and we cannot easily attain great accuracy by the truncation method.

Of course, the calculations need an electronic digital computer, even for small values of the number M of terms before truncation, since the elements of the matrix K_{st} are themselves defined by an infinite series.

However, before describing our numerical results, we bring out the mathematical implications of there being no waves in the upstream region $x < 0$. We have $\delta = y$ for $x = 0$, $0 \leqslant y \leqslant d$, so that (6.3) gives

$$\int_0^d y \sin ry\, dy + \int_0^\pi \delta(0,y) \sin ry\, dy = 0 \quad (1 \leqslant r \leqslant K). \tag{6.17}$$

It is remarkable that the flow can adjust itself to satisfy these K conditions. Indeed, if we assume that $\delta(0,y)$ is bounded *independently* of d in the interval $[d,\pi]$, we can show that when d is sufficiently close to π the conditions cannot be satisfied. In other words, we anticipate large values of δ when the gap between the top of the obstacle and the channel roof is small. Thus regions of closed streamlines, i.e. rotors, occur.

We have seen that waves, and sometimes rotors, occur downstream of the barrier. However, upstream the disturbance is a sum of terms, each of which is exponential in x. The exponential which decays least rapidly as $x \to -\infty$ is

$$\exp\{[(K+1)^2 - k^2]^{\frac{1}{2}} x\}.$$

Now $K < k < K+1$, so that we can write

$$k = K + 1 - \epsilon,$$

where $0 < \epsilon < 1$. So, for large k, the disturbance upstream tends to zero like $\exp\{[2\epsilon(K+1)]^{\frac{1}{2}} x\}$. Therefore, when the incident stream has a large k^2, the disturbance upstream is very small where $x \gg k^{-\frac{1}{2}}$. This means that streamlines are undeflected until they enter a 'boundary layer' of thickness $k^{-\frac{1}{2}}$ on the front side of the obstacle.

The computer we used could rapidly invert matrices as large as 150×150, so we took $M = 150$ in all our computations. We checked the accuracy of the coefficients A_r by computing both sides of the first of equations (6.4) for a range of values of y. If y was not close to d, the two sides of the equation differed by less than 1% for most of the cases considered. However, as y approached the value d, the discrepancy increased. This behaviour is probably due to the singularity of

† In fact, if we put

$$x^{-\frac{1}{2}} = \sum_{n=1}^{\infty} a_n \sin nx \quad (0 < x < \pi),$$

we can show that $a_n = (2/\pi n)^{\frac{1}{2}} + O(n^{-1})$ as $n \to \infty$.

$(\partial\delta/\partial y)_{x=0}$ at $y = d$, which is associated with the edge of the obstacle. The Fourier series converges very slowly near such a singularity.

Once the coefficients A_r are determined the streamlines can be computed from the relation

$$y - \delta(x, y) = y_{-\infty}, \tag{6.18}$$

where $y_{-\infty}$ is the height of the streamline at upstream infinity. In view of the results of the test applied to A_r, these streamlines will be accurate except near the tip of the obstacle.

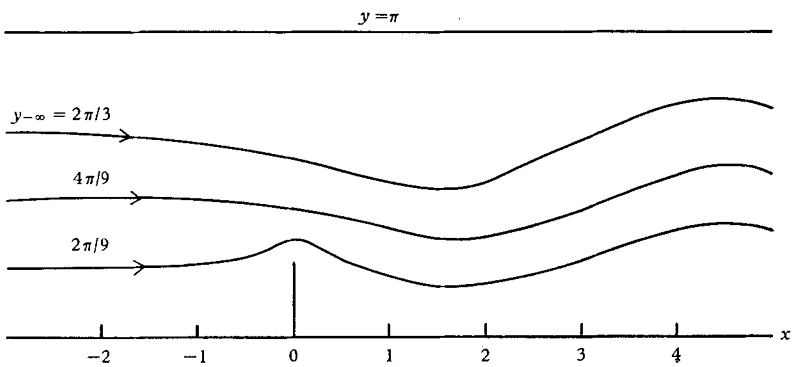

FIGURE 4. Flow past a vertical wall: $d = \frac{1}{4}\pi$, $k = 1{\cdot}5$. The drag is computed such that

$$D/\rho_{-\infty} U^2_{-\infty} = \sum_{r=1}^{K} (k^2 - r^2) A_r^2 = 0{\cdot}098 \text{ with } A_1 = 0{\cdot}279.$$

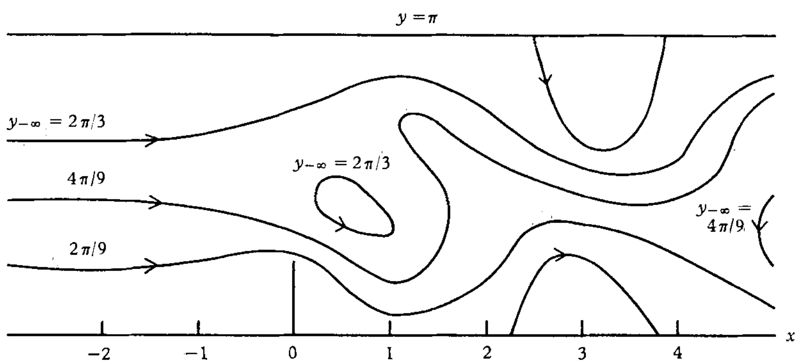

FIGURE 5. Flow past a vertical wall: $d = \frac{1}{4}\pi$, $k = 2{\cdot}5$. The drag is such that $D/\rho_{-\infty} U^2_{-\infty} = 0{\cdot}88$ with $A_1 = 0{\cdot}221$, $A_2 = 0{\cdot}526$.

The results for $d = \frac{1}{4}\pi$ and $k = 1{\cdot}5$, $k = 2{\cdot}5$, $k = 3{\cdot}5$ and $k = 4{\cdot}5$ are shown in figures 4–7. Only streamlines coming from upstream infinity are plotted, for the sake of clarity. The formation of detached rotors with strong jets threading in between them is quite striking. There is, of course, a street of rotors extending to downstream infinity in these cases. For $d = \frac{1}{2}\pi$ and $k = 1{\cdot}5$ we see from figure 8 that, as we anticipated, a rotor has formed and the jet is quite strong. For $d = \frac{1}{2}\pi$ and $k = 2{\cdot}5$ (see figure 9) the jet is very strong and winds through the rotors in a complicated way.

An unexpected feature is the separation of the flow from the bottom of the channel *ahead* of the obstacle in some cases.

For $d = \frac{1}{2}\pi$ and $k = 3{\cdot}5$ the discrepancies when the computed A_r were tested in (6.4) were typically 4%, and at $d = \frac{1}{2}\pi$ and $k = 4{\cdot}5$ they were typically 20%, so we shall not reproduce these two cases, which need a larger value of M.

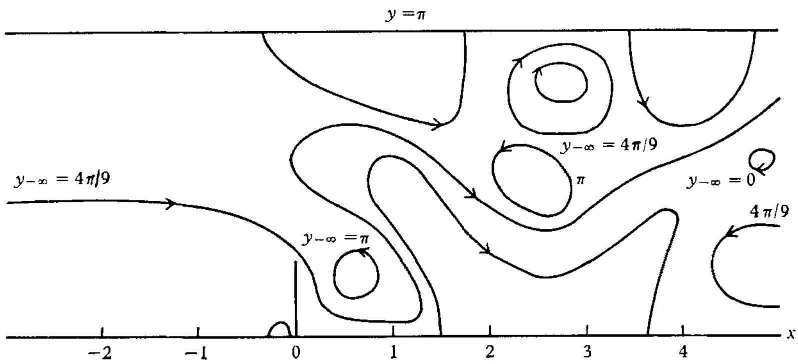

FIGURE 6. Flow past a vertical wall: $d = \frac{1}{4}\pi$, $k = 3{\cdot}5$. The drag is such that $D/\rho_{-\infty} U^2_{-\infty} = 8{\cdot}0$ with $A_1 = 0{\cdot}345$, $A_2 = 0{\cdot}641$, $A_3 = 1{\cdot}00$.

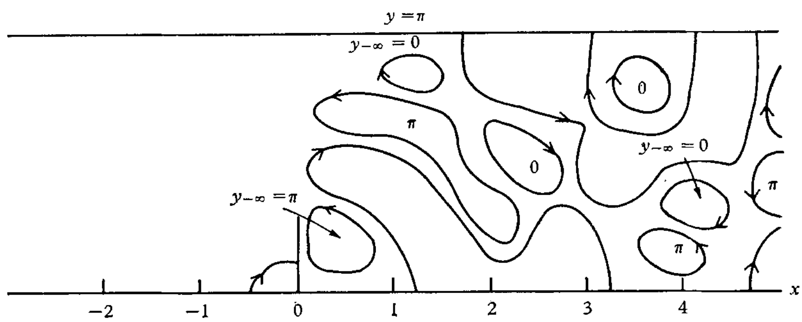

FIGURE 7. Flow past a vertical wall: $d = \frac{1}{4}\pi$, $k = 4{\cdot}5$. The drag is such that $D/\rho_{-\infty} U^2_{-\infty} = 91{\cdot}0$ with $A_1 = 0{\cdot}759$, $A_2 = 1{\cdot}35$, $A_3 = 1{\cdot}70$, $A_4 = 2{\cdot}03$.

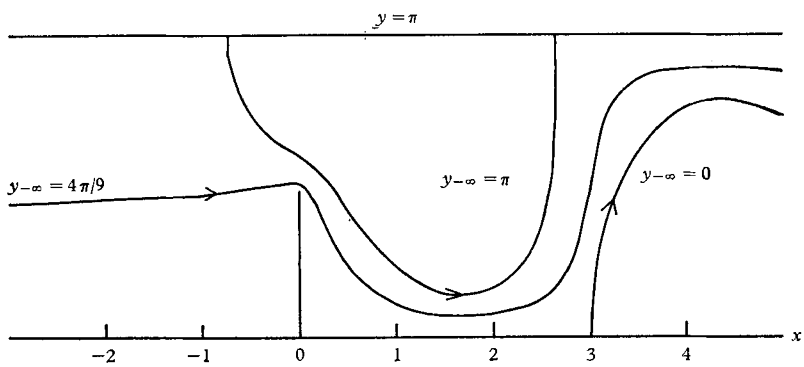

FIGURE 8. Flow past a vertical wall: $d = \frac{1}{2}\pi$, $k = 1{\cdot}5$. The drag is such that $D/\rho_{-\infty} U^2_{-\infty} = 9{\cdot}1$ with $A_1 = 2{\cdot}69$.

7. Conclusions

In general we have excluded consideration of any resonance from our work. Thus, provided that there is no resonance and that no lower bound is placed on the pressure, we believe the discussion of § 3 implies that a steady flow free of

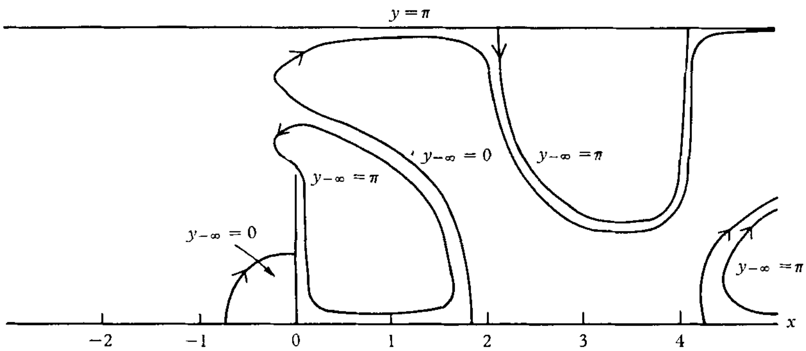

FIGURE 9. Flow past a vertical wall: $d = \frac{1}{2}\pi$, $k = 2{\cdot}5$. The drag is such that $D/\pi\rho_{-\infty}U^2_{-\infty} = 2100$ with $A_1 = 14{\cdot}8$, $A_2 = 20{\cdot}0$.

upstream waves exists in the channel with a rigid lid, whatever the internal Froude number and size of the obstacle. In short, we find that there is no critical value of k^2 for blocking. Blocking seems to be always associated with a pressure (or, equivalently, an energy) restriction and not with the non-existence of any steady solution.

Also in §3 we pointed to a possible omission in Long's argument for a critical internal Froude number. However, a recent paper by Mrs K. Trustrum (1964) supports Long's conclusion. She linearized the equations of unsteady flow, replacing the inertial term $\mathbf{u}.\nabla\mathbf{u}$ by an 'Oseen' advection term $U\mathbf{i}.\nabla\mathbf{u}$, where U is a constant. This seems a reasonable procedure, and Mrs Trustrum points out that it is rigorously valid for slightly porous obstacles. This mathematical simplification enables unsteady flows to be examined so that, by starting the flow from rest in an initial-value problem, the nature of the flow far upstream can be deduced.

The vertical dependence of the velocity field is represented as a Fourier integral, and the evolution in time of a single component is examined. Its fate is shown to depend on its component internal Froude number, $g\beta\lambda^2/4\pi^2u^2$ where λ is the wavelength of the y-dependence. Analysis of the Laplace transform of the equations of motion show that if this component number is greater than unity the corresponding Fourier component does not tend to zero at upstream infinity —that is to say, it blocks the flow.

In the next stage of her argument, Mrs Trustrum's model diverges from ours. Mathematical difficulties prevent the treatment of unsteady flow over an obstacle which fills only part of the channel, so the velocity distribution is specified on the entire cross-section $x = 0$ instead. In general this distribution will involve all Fourier components, and therefore all wavelengths λ will arise. Thus a disturbance of the flow at $x = -\infty$ will remain as $t\to\infty$ for any value of the overall Richardson number—an even stronger conclusion than Long's. However, for a distribution of horizontal dipoles on the cross-section $x = 0$ this upstream influence vanishes. Indeed, our solution §5 for the dipole can come directly from Mrs Trustrum's analysis. Also, her inverse method gives no streamline, other than $y = 0, \pi$, fixed for all time.

When the domain of flow is the fixed half-strip ($x \leqslant 0$, $0 \leqslant y \leqslant \pi$, $t \geqslant 0$) and the velocity is physically specified on the whole cross-section $x = 0$ this argument is applicable, and Mrs Trustrum obtained qualitative agreement with Debler's (1961) experiments on flow into a line sink in a semi-infinite closed channel. However, for a partly constricted infinite closed channel our analysis suggests that the flow adjusts itself to remove those Fourier components which do not decay as $x \to -\infty$. This is seen most clearly in the case of the straight barrier in § 6. The Fourier components which do not decay as $x \to -\infty$ are proportional to $\sin ny$ $(1 \leqslant n \leqslant K)$. The boundary conditions on the barrier imply that $\delta(0, y)$ has the values y on the wall $(0 \leqslant y \leqslant d)$ but its values in the gap $(d < y < \pi)$ are subject only to matching requirements. We have constructed solutions $\delta(x, y)$ which decay exponentially as $x \to -\infty$, and this means that the value of δ in the gap has adjusted itself so that $\delta(0, y)$ is orthogonal to $\sin ny$ in the whole interval $[0, \pi]$. This freedom of $\delta(0, y)$ in the gap is crucial, and we suggest that Mrs Trustrum's conclusions may not hold for flow past a partial constriction.

It is a pleasure to thank Professor D. S. Jones for his advice on diffraction theory; Mr L. E. Fraenkel for suggesting that we write § 5; Professor M. H. Rogers for his guidance and Miss C. Faithfull and Miss P. Virgo for their help with the numerical computation of §§ 5, 6; and Mr B. Goldstein of the Institute for Space Studies for his skilful programming of the bulk of the computations of § 6.

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